

A master identity for homotopy Gerstenhaber algebras

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Abstract

We produce a master identity $\{\tilde{m}\}\{\tilde{m}\} = 0$ for homotopy Gerstenhaber algebras, as defined by Getzler and Jones and utilized by Kimura, Voronov, and Zuckerman in the context of topological conformal field theories. To this end, we introduce the notion of a “partitioned multilinear map” and explain the mechanics of composing such maps. In addition, many new examples of pre-Lie algebras and homotopy Gerstenhaber algebras are given.

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1 Introduction

The proliferation of algebraic identities in string theory, such as the lower identities for BRST complexes and topological vertex operator algebras (TVOA) in Lian and Zuckerman’s leading work [16] (see also [21])

and those given by Kimura, Voronov, and Zuckerman for homotopy Gerstenhaber algebras -specifically for TVOA's- in [13], has led the author to study a common language for multilinear maps and their compositions. Gerstenhaber's braces

$$x\{y\} = x \circ y$$

(which we will write as $\{x\}\{y\}$, following the notation of [13]) have been in existence for more than thirty years and denote a certain rule of composition of two multilinear maps x, y on a vector space A with values in A . They were extended by Getzler in [10] to the composition

$$x\{y_1, \dots, y_n\} = \{x\}\{y_1, \dots, y_n\}$$

of several multilinear maps with one. These composition rules were essential in redefining, and exploring new properties and examples of, commonly studied algebras like associative, Lie, homotopy associative (A_∞), homotopy Lie (L_∞), Batalin-Vilkovisky, Gerstenhaber, and homotopy Gerstenhaber (G_∞) algebras. For example, many fundamental (sets of) identities, most notably associativity, can be grouped under

$$m \circ m = 0,$$

where m might be a homogeneous multilinear map or a formal infinite sum of such maps. An overview will be given in Section 2.1.

In turn, the iterated composition operations have been shown to satisfy many algebraic identities (see [22]) resembling those which arise in the context of operads and homotopy algebras. Curiously, the multibraces

$$\{v_1, \dots, v_m\} \cdots \{w_1, \dots, w_n\}$$

of [13], defined on a TVOA or any G_∞ algebra, satisfy similar (but more numerous) identities. As a first step in understanding the similarities, the Gerstenhaber-Getzler braces were extended by the author in [1] to multiple substitutions/compositions of the form

$$\{x\}\{y_1, \dots, y_m\} \cdots \{z_1, \dots, z_n\}$$

on the Hochschild complex

$$C^\bullet(A) = \text{Hom}(TA; A)$$

of multilinear maps on A , and eventually to

$$\{x_1, \dots, x_m\} \cdots \{y_1, \dots, y_n\}$$

on

$$C^{\bullet, \bullet}(A) = \text{Hom}(TA; TA)$$

(examples can be found in Section 2.1). Although many new results and identities were unveiled, the “unadorned” multilinear maps were apparently not enough to describe the rich structure in [13] in a nutshell.

In this paper we introduce “partitioned multilinear maps” $x(\pi)$ which are no different from ordinary multilinear maps except in composition. The ordered partitions

$$\pi = (i_1 | \cdots | i_r)$$

of nonnegative integers $i_1 + \cdots + i_r$ give us a grouping of the arguments of the map x , and the algebra generated by the “products” of such partitions governs the composition rule: if

$$\pi * \pi' = \pi_1 + \cdots + \pi_n,$$

then

$$\{x(\pi)\}\{y(\pi')\} = \sum_{i=1}^n z_i(\pi_i),$$

where the resulting partitioned maps $z_i(\pi_i)$ are rigorously defined in Sections 2.3.1 and 2.3.5. The product $*$ is reduced to the familiar rule

$$(i) * (j) = (i + j - 1)$$

for singletons, which says that

$$\{x(i)\}\{y(j)\} = z(i + j - 1),$$

or the composition of i -linear and j -linear maps is an $(i + j - 1)$ -linear map. With this notation, it is possible to write the algebraic master equation for homotopy Gerstenhaber algebras (which were previously defined only in terms of a topological operad) as

$$\{\tilde{m}\}\{\tilde{m}\} = 0,$$

where

$$m = \sum_{\pi} m(\pi)$$

is a formal sum of all the partitioned multilinear maps involved, and \tilde{m} is a term-by-term modification of m (only in \pm signs). This identity is to be interpreted as follows: the finite sum of multilinear maps of “type” $\tilde{\pi}$, which result from the compositions of all $m(\pi)$ and $m(\pi')$ such that

$$\pi * \pi' = \tilde{\pi} + \dots,$$

is identically zero for all ordered partitions $\tilde{\pi}$. If only singletons (i) are allowed as valid partitions, we obtain an A_{∞} algebra. Some interesting subalgebras of the G_{∞} algebra are studied in Section 4.1. A word of caution: the master identity may be modified as

$$\{\tilde{m}\}\{\tilde{m}\} + F(\tilde{m}) = 0;$$

see Remark 7.

In addition to the master identity, which frees us from the necessity of drawing several pictures for every given partition and supplies us with an algorithm for writing the subidentities, we will introduce a variety of new examples of homotopy Gerstenhaber algebras and pre-Lie algebras. For example, the algebra \mathcal{P} of regular partitions is a right pre-Lie algebra, any vertex operator algebra with the Wick product is a left pre-Lie algebra, and it may be possible to build up a G_{∞} algebra from scratch starting with a square-zero Batalin-Vilkovisky type operator and the Φ -operators introduced by the author in [2]. The multibraces notation of [13] will be replaced by the partitioned map notation $m(i_1 | \dots | i_r)$ so that braces can be used to denote composition.

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2 Partitioned maps

2.1 Multibraces

This is a short review of the multibraces notation introduced by Gerstenhaber [9] and further developed by Getzler [10] and the author [1]. For more details and historical references the reader is referred to [1].

For the time being it suffices to consider compositions of multilinear maps

$$x : A^{\otimes n} \rightarrow A$$

where A is a (super) graded vector space

$$A = \oplus_{n \in \mathbb{Z}} A^n.$$

These maps live in the **Hochschild complex**

$$C^\bullet(A) = \oplus_{n \geq 0} \text{Hom}(A^{\otimes n}; A),$$

which may be replaced by its “completion”

$$C^\bullet(A) = \text{Hom}(TA; A)$$

to accommodate formal infinite sums (TA is the tensor algebra on A). Elements of A , or maps from the (characteristic zero) field into A , are treated on the same footing as higher multilinear maps. If x is an n -linear map ($n \geq 0$) which changes the total super degree of its arguments by the integer amount s , it is assigned the gradings

$$d(x) = n - 1 \quad \text{and} \quad |x| = s.$$

The compact expression

$$\{x\}\{y_1, \dots, y_k\} \cdots \{z_1, \dots, z_l\}\{a_1, \dots, a_n\}, \quad (1)$$

where x, y_i, z_j are maps and $a_1, \dots, a_n \in A$, indicates that:

- Every symbol except x is to be substituted into one on the left, in every possible way. But:
- Symbols within the same pair of braces cannot be substituted into one another, and the internal order within each pair must be retained. Then:
- All such expressions are to be added up, each term accompanied by a product of signs

$$(-1)^{d(u)d(v)+|u||v|}$$

whenever two symbols u, v are interchanged with respect to (1).

Super and d -degrees are preserved under such multiple composition-substitutions.

Example 1 Let $d(x) = 2$, $d(y) = 1$, and $d(z) = 2$. Then

$$\begin{aligned} & \{x\}\{y, z\}\{a, b, c, d, e, f\} \\ = & (-1)^{d(z)(d(a)+d(b))+|z|(|a|+|b|)} x(y(a, b), z(c, d, e), f) \\ & + (-1)^{d(z)(d(a)+d(b)+d(c))+|z|(|a|+|b|+|c|)} x(y(a, b), c, z(d, e, f)) \\ & + (-1)^{d(y)d(a)+|y||a|+d(z)(d(a)+d(b)+d(c))+|z|(|a|+|b|+|c|)} x(a, y(b, c), z(d, e, f)) \\ = & (-1)^{|z|(|a|+|b|)} x(y(a, b), z(c, d, e), f) \\ & + (-1)^{|z|(|a|+|b|+|c|)} x(y(a, b), c, z(d, e, f)) \\ & - (-1)^{|y||a|+|z|(|a|+|b|+|c|)} x(a, y(b, c), z(d, e, f)). \end{aligned}$$

But note that

$$\begin{aligned} & \{x\}\{y\}\{z\}\{a, b, c, d, e, f\} \\ = & \pm x(y(a, b), z(c, d, e), f) \pm x(y(a, b), c, z(d, e, f)) \pm x(a, y(b, c), z(d, e, f)) \\ & \pm x(z(a, b, c), y(d, e), f) \pm x(z(a, b, c), d, y(e, f)) \pm x(a, z(b, c, d), y(e, f)) \\ & \pm x(y(z(a, b, c), d), e, f) \pm x(y(a, z(b, c, d)), e, f) \pm x(a, y(z(b, c, d), e), f) \\ & \pm x(a, y(b, z(c, d, e)), f) \pm x(a, b, y(z(c, d, e), f)) \pm x(a, b, y(c, z(d, e, f))). \end{aligned}$$

Example 2 By definition,

$$\{x\}\{a_1\} \cdots \{a_n\} = \sum_{\sigma \in S_n} (-1)^{\text{sgn}(\sigma)} x(a_{\sigma(1)}, \dots, a_{\sigma(n)})$$

(super degrees ignored for simplicity).

Example 3 If m is an even bilinear map, the condition

$$\{m\}\{m\} = 0$$

is equivalent to associativity:

$$\begin{aligned} & \{m\}\{m\}\{a, b, c\} \\ &= m(m(a, b), c) + (-1)^{d(m)d(a)+|m||a|} m(a, m(b, c)) \\ &= m(m(a, b), c) - m(a, m(b, c)) = 0. \end{aligned}$$

Example 4 For an even bilinear associative map m , the bracket defined by

$$[a, b]_m = \{m\}\{a\}\{b\} = m(a, b) - (-1)^{|a||b|} m(b, a)$$

is a (graded) Lie bracket, because

$$\{m\}\{m\}\{a, b, c\} = 0 \quad \forall a, b, c$$

implies that

$$\{m\}\{m\}\{a\}\{b\}\{c\} = 0 \quad \forall a, b, c,$$

which is equivalent to the Jacobi identity for $[\ , \]_m$.

Example 5 A vector space A with a bilinear multiplication map

$$m(a, b) = a \star b$$

is a **right pre-Lie algebra** if the identity

$$(a \star b) \star c - a \star (b \star c) = (-1)^{|b||c|} ((a \star c) \star b - a \star (c \star b)) \quad (2)$$

holds (there may be any number of gradings on A). Gerstenhaber [9] showed that his bigraded composition product

$$x \circ y = \{x\}\{y\}$$

on $-$ -truncated- $C^\bullet(A)$ is a right pre-Lie product, and therefore the bigraded **Gerstenhaber bracket**

$$[x, y] \stackrel{\text{def}}{=} \{x\}\{y\} - (-1)^{d(x)d(y)+|x||y|} \{y\}\{x\}$$

leads to a bigraded Lie algebra. This last statement is easy to see in general for (A, m) because (2) can be expressed as

$$\{m\}\{m\}\{a, \{b\}\{c\}\} = 0 \quad \forall a, b, c,$$

which implies

$$\{m\}\{m\}\{a\}\{b\}\{c\} = 0 \quad \forall a, b, c.$$

Additional results about $C^\bullet(A)$ can be found in Sections 2.3.4 and 4.2.

Example 6 In order to write master identities for **homotopy associative and homotopy Lie** (A_∞ and L_∞) **algebras**, we are forced to introduce the single grading

$$||x|| = d(x) + |x|$$

on $C^\bullet(A)$, and to define

$$\tilde{x}(a_1, \dots, a_n) = (-1)^{(n-1)||a_1|| + (n-2)||a_2|| + \dots + ||a_{n-1}||} \{x\} \{a_1, \dots, a_n\}$$

for $d(x) = n - 1$. Then, for example, an A_∞ algebra is nothing but a graded vector space A with a formal infinite sum of maps

$$m = m(1) + m(2) + \dots$$

satisfying

$$\{\tilde{m}\}\{\tilde{m}\} = 0.$$

In this setting we have

$$d(m(i)) = i - 1 \quad \text{and} \quad |m(i)| \equiv i \pmod{2},$$

so that $||m(i)||$ is always odd. Moreover, an interchange of u and v results in the sign

$$(-1)^{||u|| ||v||}.$$

The master identity above reduces to the associativity condition on $m(2)$ when $m = m(2)$.

Example 7 It is well-known [15] that graded antisymmetrization of the products in an A_∞ algebra leads to an L_∞ algebra. The proof in [1] uses the following simple fact: the usual L_∞ identities are equivalent to

$$\{\tilde{m}\}\{\tilde{m}\}\{a_1\} \cdots \{a_n\} = 0 \quad \forall a_1, \dots, a_n$$

in this case, which is an immediate consequence of

$$\{\tilde{m}\}\{\tilde{m}\}\{a_1, \dots, a_n\} = 0 \quad \forall a_1, \dots, a_n.$$

Example 8 In [1], the Hochschild complex was extended to

$$C^{\bullet, \bullet}(A) = \text{Hom}(TA; TA)$$

by allowing multilinear maps with values in TA , including elements of TA . The rules remain the same, but many formerly inadmissible expressions have a meaning in the new complex. For example,

$$\{a_1, \dots, a_n\} \stackrel{\text{def}}{=} a_1 \otimes \cdots \otimes a_n,$$

and

$$\{a\}\{b\} = \{a, b\} \pm \{b, a\} = a \otimes b - (-1)^{|a||b|} b \otimes a.$$

2.2 The algebra of ordered partitions

2.2.1 Regular partitions

We will define a binary multiplication $*$ on the free abelian group \mathcal{P} spanned by the **(regular) ordered partitions**

$$\pi = (i_1 | \cdots | i_r) \quad r \geq 1, \quad i_1, \dots, i_r \geq 1$$

of all positive integers. These partitions are to be thought of as grouping the arguments of multilinear maps

$$x : A^{\otimes(i_1 + \cdots + i_r)} \rightarrow A$$

of **type** π . We grade the basis elements π of \mathcal{P} by

$$d(\pi) = i_1 + \cdots + i_r - 1$$

(total number of **arguments** minus one) and

$$\bar{d}(\pi) = r - 1$$

(total number of **slots** minus one), and remark that both degrees will be preserved under multiplication. The product $\pi * \pi'$ of two ordered partitions will be a sum

$$\pi_1 + \cdots + \pi_n$$

of basis elements, where the multiplicity of each π_l is chosen to be 1 for convenience. We will later interpret the outcome as the collection of types of multilinear maps that arise from the composition of two maps of types π and π' . An arbitrary element of the algebra \mathcal{P} will be denoted by p , \hat{p} , etc. **Notation:** If π appears with a nonzero (integral) coefficient in p , we will write $\pi \rightarrow p$.

We first describe the product

$$p = (i) * (j_1 | \cdots | j_s)$$

with

$$d(p) = i + j_1 + \cdots + j_s - 2, \quad \bar{d}(p) = s - 1$$

by

$$p = \sum_{u_1 + \cdots + u_s = i-1, u_l \geq 0} (j_1 + u_1 | \cdots | j_s + u_s). \quad (3)$$

The most general definition of

$$p = \pi * \pi' = (i_1 | \cdots | i_r) * (j_1 | \cdots | j_s)$$

is first approximated by

$$\hat{p} = \sum_{l=1}^r \sum_{\tilde{\pi} \rightarrow (i_l) * \pi'} (i_1 | \cdots | i_{l-1} | \tilde{\pi} | i_{l+1} | \cdots | i_r), \quad (4)$$

after which we combine like terms under coefficient 1 and write

$$p = \sum_{\tilde{\pi} \rightarrow \hat{p}} \tilde{\pi}, \quad (5)$$

that is, we multiply one slot of π with π' at a time and insert the resulting basis elements into the slot in question one by one, finally erasing duplicate terms.

Example 9 *In the following computation, we reduce the coefficient of $(1|1|3|4)$ to 1 although it shows up twice in \hat{p} :*

$$(1|1|4) * (1|3) = (1|3|1|4) + (1|1|3|4) + (1|1|4|3) + (1|1|2|5) + (1|1|1|6).$$

Example 10 *The special case*

$$(i) * (j) = (i + j - 1) = (j) * (i)$$

reflects the fact that the composition of an i -linear map with a j -linear map in the sense of Gerstenhaber is an $(i + j - 1)$ -linear map.

The partition (1) acts as a two-sided identity in the noncommutative algebra \mathcal{P} . The product $*$ is not associative nor left pre-Lie, but it does turn out to be right pre-Lie. These statements are false if we count multiplicities.

Proposition 1 *The algebra \mathcal{P} of ordered partitions is a right pre-Lie algebra under the product $*$, namely, the identity*

$$(p_1 * p_2) * p_3 - p_1 * (p_2 * p_3) = (p_1 * p_3) * p_2 - p_1 * (p_3 * p_2)$$

holds. As a result, the (ungraded) Gerstenhaber bracket

$$[p_1, p_2] \stackrel{\text{def}}{=} p_1 * p_2 - p_2 * p_1$$

on \mathcal{P} is a Lie bracket.

Proof. It suffices to give a proof for basis elements

$$\pi_1 = (i_1 | \cdots | i_r), \quad \pi_2 = (j_1 | \cdots | j_s), \quad \pi_3 = (k_1 | \cdots | k_t).$$

(A) Special case: we will show that when $\pi_1 = (i)$ associativity holds on the nose, i.e.

$$((i) * \pi_2) * \pi_3 = (i) * (\pi_2 * \pi_3).$$

Let us compare the two sides (up to repetition of terms, hence the symbol \approx instead of $=$) from the definition. We have

$$(i) * \pi_2 = \sum_{u_1 + \cdots + u_s = i-1} (j_1 + u_1 | \cdots | j_s + u_s)$$

and

$$\begin{aligned} & ((i) * \pi_2) * \pi_3 \\ \approx & \sum_{\alpha=1}^s \sum_{\beta=1}^t \sum_{u_1 + \cdots + u_s = i-1} \sum_{v_1 + \cdots + v_t = j_\alpha + u_\alpha - 1} (j_1 + u_1 | \cdots | j_{\alpha-1} + u_{\alpha-1} | k_1 + v_1 | \cdots | k_t + v_t | j_{\alpha+1} + u_{\alpha+1} | \cdots | j_s + u_s). \end{aligned} \tag{6}$$

On the other hand, we have

$$\begin{aligned} & \pi_2 * \pi_3 \\ \approx & \sum_{\alpha=1}^s (j_1 | \cdots | (j_\alpha) * \pi_3 | \cdots | j_s) \\ \approx & \sum_{\alpha=1}^s \sum_{\beta=1}^t \sum_{w_1 + \cdots + w_t = j_\alpha - 1} (j_1 | \cdots | j_{\alpha-1} | k_1 + w_1 | \cdots | k_t + w_t | j_{\alpha+1} | \cdots | j_s) \end{aligned}$$

and

$$\begin{aligned} & (i) * (\pi_2 * \pi_3) \\ \approx & \sum_{\alpha=1}^s \sum_{\beta=1}^t \sum_{w_1 + \cdots + w_t = j_\alpha - 1} \sum_{z_1 + \cdots + z_{s+t-1} = i-1} (j_1 + z_1 | \cdots | j_{\alpha-1} + z_{\alpha-1} | k_1 + w_1 + z_\alpha | \cdots | k_t + w_t + z_{\alpha+t-1} | j_{\alpha+1} + z_{\alpha+t} | \cdots | j_s + z_{s+t-1}). \end{aligned} \tag{7}$$

After making a (not necessarily invertible) change of variables

$$\begin{aligned} u_1 &= z_1, \dots, u_{\alpha-1} = z_{\alpha-1} \\ u_\alpha &= z_\alpha + \cdots + z_{\alpha+t-1} \\ u_{\alpha+1} &= z_{\alpha+t}, \dots, u_s = z_{s+t-1} \\ v_1 &= w_1 + z_\alpha, \dots, v_t = w_t + z_{\alpha+t-1}, \end{aligned}$$

we obtain (6) from (7). Note that any excess terms are duplicates and can be discarded.

(B) General case: we have

$$\pi_1 * \pi_2 \approx \sum_{m=1}^r (i_1 | \cdots | (i_m) * \pi_2 | \cdots | i_r)$$

and

$$\begin{aligned} & (\pi_1 * \pi_2) * \pi_3 \\ \approx & \sum_{m=1}^r \sum_{\alpha=1}^{m-1} (i_1 | \cdots | (i_\alpha) * \pi_3 | \cdots | (i_m) * \pi_2 | \cdots | i_r) \\ & + \sum_{m=1}^r (i_1 | \cdots | ((i_m) * \pi_2) * \pi_3 | \cdots | i_r) \\ & + \sum_{m=1}^r \sum_{\alpha=m+1}^r (i_1 | \cdots | (i_m) * \pi_2 | \cdots | (i_\alpha) * \pi_3 | \cdots | i_r). \end{aligned}$$

But then $\pi_1 * (\pi_2 * \pi_3)$ is just

$$\pi_1 * (\pi_2 * \pi_3) = \sum_{m=1}^r (i_1 | \cdots | (i_m) * (\pi_2 * \pi_3) | \cdots | i_r),$$

and

$$(\pi_1 * \pi_2) * \pi_3 - \pi_1 * (\pi_2 * \pi_3) \tag{8}$$

consists of all terms in which π_2 and π_3 (in any order!) are multiplied on the right by two *distinct* slots in π_1 , thanks to the special case. Obviously,

$$(\pi_1 * \pi_3) * \pi_2 - \pi_1 * (\pi_3 * \pi_2) \tag{9}$$

consists of the exact same terms by symmetry, and the difference of (8) and (9) is zero. \square

Remark 1 *Many right pre-Lie proofs follow the same pattern: one shows*

$$(a \star b) \star c - a \star (b \star c)$$

is (possibly graded) symmetric in b and c .

2.2.2 Partitions involving zeros

In order to write down the G_∞ identities in a uniform fashion, we will need to deal with partitions

$$\pi = (i_1 | \cdots | i_r)$$

with $i_1, \dots, i_r \geq 0$. Let us denote the basis of \mathcal{P} consisting of the regular partitions by \mathcal{B} , the set of partitions with at least one zero slot by \mathcal{B}_0 , their union $\mathcal{B} \cup \mathcal{B}_0$ by $\bar{\mathcal{B}}$, and the free abelian group spanned by $\bar{\mathcal{B}}$ by $\bar{\mathcal{P}}$. We extend the multiplication $*$ to $\bar{\mathcal{P}}$ by the same rules (3)-(5). In effect, everything is as before except when there is a zero slot on the *left*: we have

$$(0) * (j_1 | \cdots | j_s) = 0$$

since there are no nonzero numbers u_1, \dots, u_s adding up to -1 (an empty sum is zero).

Example 11

$$\begin{aligned} (1|0|3) * (2) &= (2|0|3) + (1|0|4) \\ (2) * (1|0|3) &= (1|0|4) + (1|1|3) + (2|0|3). \end{aligned}$$

Remark 2 The partition (1) still acts as the left identity, but the algebra $(\bar{\mathcal{P}}, *)$ is not right pre-Lie any more. For example,

$$((i) * (0)) * (j) - (i) * ((0) * (j)) = (i + j - 2),$$

whereas

$$((i) * (j)) * (0) - (i) * ((j) * (0)) = (i + j - 2) - (i + j - 2) = 0$$

for $i \geq 2, j \geq 1$.

2.3 Partitioned multilinear maps

2.3.1 Regular partitioned maps

We introduce the notion of a **(regular) partitioned multilinear map**

$$x(\pi) = x(i_1 | \dots | i_r) : A^{\otimes i_1} \otimes \dots \otimes A^{\otimes i_r} \rightarrow A, \quad \pi \in \mathcal{B}$$

as an $(i_1 + \dots + i_r)$ -linear map which is labelled by a regular ordered partition π and is distinguished from ordinary maps only by the way we compose it (Getzler and Jones mention multilinear maps $m_{k,l}$ which are similar to our $m(k|l)$ in [12] but do not elaborate on their composition properties). For substitution into $x(\pi)$, we will use the notation

$$\begin{aligned} & \{x(i_1 | \dots | i_r)\} \{a_1, \dots, a_{i_1} | a_{i_1+1}, \dots, a_{i_1+i_2} | \dots | a_{i_1+\dots+i_{r-1}+1}, \dots, a_{i_1+\dots+i_r}\} \\ = & \{x(i_1 | \dots | i_r)\} \{a^{(1)} | \dots | a^{(r)}\} \end{aligned}$$

rather than

$$\{x(i_1 | \dots | i_r)\} \{a_1, \dots, a_{i_1+\dots+i_r}\}.$$

When

$$x = y_1(\pi_1) + \dots + y_k(\pi_k) \quad (\text{with } d(\pi_1) = \dots = d(\pi_k) = n - 1)$$

is a sum of maps of different types, we will still write

$$\{x\} \{a_1, \dots, a_n\}$$

to mean

$$\{y_1(\pi_1)\} \{a^{(1)} | \dots | a^{(r)}\} + \dots \quad (\text{say for } \pi_1 = (i_1 | \dots | i_r), \dots),$$

where each map $y_l(\pi_l)$ is followed by the suitable barred braces. In particular, we may use ordinary braces after the composition of two maps $x(\pi)$ and $y(\pi')$ to denote the sum of all cases. Recalling from the previous section that $\{x(\pi)\} \{y(\pi')\}$ is designed to be of the form

$$\{x(\pi)\} \{y(\pi')\} = \sum_{\tilde{\pi} \rightarrow \pi * \pi'} z_{\tilde{\pi}}(\tilde{\pi}),$$

we proceed to define each component $z_{\tilde{\pi}}(\tilde{\pi})$ rigorously. In the following account, the \pm signs preceding each term are computed as in Section 2.1, depending solely on the super degrees and the d -degrees of the maps and elements involved; we will in general omit the full expressions to save space. Note that

$$d(x(\pi)) = d(\pi) \quad \text{and} \quad \bar{d}(x(\pi)) = \bar{d}(\pi)$$

by definition.

As with partitions, we start with the simpler case

$$\{x(i)\} \{y(\pi')\}, \quad \pi' = (j_1 | \dots | j_s).$$

Let

$$\tilde{\pi} \rightarrow (i) * (\pi'),$$

with

$$\tilde{\pi} = (j_1 + u_1 | \cdots | j_s + u_s) = (k_1 | \cdots | k_s), \quad u_1 + \cdots + u_s = i - 1$$

(note that all such $\tilde{\pi}$ are necessarily distinct). To define

$$\{z_{\tilde{\pi}}(\tilde{\pi})\} \{a^{(1)} | \cdots | a^{(s)}\},$$

we consider all possible subdivisions S

$$\begin{aligned} \{a^{(1)}\} &= \{b^{(1)}, c^{(1)}, d^{(1)}\} \\ &\vdots \\ \{a^{(s)}\} &= \{b^{(s)}, c^{(s)}, d^{(s)}\} \end{aligned} \tag{10}$$

of the elements

$$\{a_1, \dots, a_{k_1 + \dots + k_s}\} = \{a^{(1)} | \cdots | a^{(s)}\}$$

such that $c^{(1)}$ contains j_1 consecutive elements in $a^{(1)}$, $c^{(2)}$ has j_2 in $a^{(2)}$, and so on. Then we have

$$\begin{aligned} &\{z_{\tilde{\pi}}(\tilde{\pi})\} \{a^{(1)} | \cdots | a^{(s)}\} \\ \stackrel{\text{def}}{=} &\sum_S \pm \{x(i)\} \{ \{b^{(1)}\} \cdots \{b^{(s)}\}, \{y(j_1 | \cdots | j_s)\} \{c^{(1)} | \cdots | c^{(s)}\}, \{d^{(1)}\} \cdots \{d^{(s)}\} \}. \end{aligned}$$

The notation $\{b^{(1)}\} \cdots \{b^{(s)}\}$, for example, indicates that we sum over all possible permutations of the remaining elements that come *before* the chosen strings $c^{(l)}$, while retaining the order within each $b^{(l)}$. The strings $b^{(l)}$ and $d^{(l)}$ may be empty.

Example 12 *Since*

$$(3) * (2|4) = (2|6) + (3|5) + (4|4),$$

the composition $\{x(3)\}\{y(2|4)\}$ is a sum of three partitioned maps. The first one is given by

$$\begin{aligned} &\{x(3)\}\{y(2|4)\}\{a, b|c, d, e, f, g, h\} \\ = &\pm \{x(3)\}\{ \{y(2|4)\}\{a, b|c, d, e, f\}, g, h\} \\ &\pm \{x(3)\}\{c, \{y(2|4)\}\{a, b|d, e, f, g\}, h\} \\ &\pm \{x(3)\}\{c, d, \{y(2|4)\}\{a, b|e, f, g, h\} \}, \end{aligned}$$

the second one by

$$\begin{aligned} &\{x(3)\}\{y(2|4)\}\{a, b, c|d, e, f, g, h\} \\ = &\pm \{x(3)\}\{ \{y(2|4)\}\{a, b|d, e, f, g\}, c, h\} \\ &\pm \{x(3)\}\{ \{y(2|4)\}\{a, b|d, e, f, g\}, h, c\} \\ &\pm \{x(3)\}\{d, \{y(2|4)\}\{a, b|e, f, g, h\}, c\} \\ &\pm \{x(3)\}\{a, \{y(2|4)\}\{b, c|d, e, f, g\}, h\} \\ &\pm \{x(3)\}\{a, d, \{y(2|4)\}\{b, c|e, f, g, h\} \} \\ &\pm \{x(3)\}\{d, a, \{y(2|4)\}\{b, c|e, f, g, h\} \}, \end{aligned}$$

and the third one by

$$\begin{aligned} &\{x(3)\}\{y(2|4)\}\{a, b, c, d|e, f, g, h\} \\ = &\pm \{x(3)\}\{ \{y(2|4)\}\{a, b|e, f, g, h\}, c, d\} \\ &\pm \{x(3)\}\{a, \{y(2|4)\}\{b, c|e, f, g, h\}, d\} \\ &\pm \{x(3)\}\{a, b, \{y(2|4)\}\{c, d|e, f, g, h\} \}. \end{aligned}$$

Remark 3 Note that unlike the ordinary composition for non-partitioned maps, the order of the elements $\{a^{(1)}|\dots|a^{(s)}\}$ do change in the final substitution. However, the order in each slot $a^{(l)}$ remains the same.

In order to extend the definition to

$$\{x(\pi)\}\{y(\pi')\} = \{x(i_1|\dots|i_r)\}\{y(j_1|\dots|j_s)\},$$

we first identify the slot(s) in $(i_1|\dots|i_r)$ which give rise to the partition $\tilde{\pi}$ in the product $\pi * \pi'$ (when $r \geq 2$, it is possible to have the outcome $\tilde{\pi}$ repeated in the multiplication process; we then reduce the coefficient of $\tilde{\pi}$ in $\pi * \pi'$ to 1 and add all the multilinear maps of type $\tilde{\pi}$ in $\{x(\pi)\}\{y(\pi')\}$). Without loss of generality, assume that the culprit is the first slot and that $\tilde{\pi}$ is not repeated. Then

$$\tilde{\pi} = (j_1 + k_1|\dots|j_s + k_s|i_2|\dots|i_r),$$

where

$$k_1 + \dots + k_s = i_1 - 1,$$

and

$$\begin{aligned} & \{z_{\tilde{\pi}}(\tilde{\pi})\}\{a^{(1)}|\dots|a^{(s+r-1)}\} \\ \stackrel{\text{def}}{=} & \pm\{\{x(i_1|\dots|i_r)\}\{-|a^{(s+1)}|\dots|a^{(s+r-1)}\}\}\{y(j_1|\dots|j_s)\}\{a^{(1)}|\dots|a^{(s)}\}. \end{aligned}$$

In other words, we fix the arguments of the remaining $r - 1$ slots of $x(\pi)$, and compose the two maps as if the first is an ordinary one with $\bar{d} = 0$, according to the recipe in the previous paragraph.

Example 13 We verify that

$$(1|2) * (2|3) = (2|3|2) + (1|2|4) + (1|3|3),$$

and compute the first partitioned map which arises from multiplying the first slot in $(1|2)$ by $(2|3)$:

$$\begin{aligned} & \{x(1|2)\}\{y(2|3)\}\{a,b|c,d,e|f,g\} \\ = & \pm\{\{x(1|2)\}\{-|f,g\}\}\{y(2|3)\}\{a,b|c,d,e\}. \end{aligned}$$

The last two maps come from multiplication with the second slot, hence we have

$$\begin{aligned} & \{x(1|2)\}\{y(2|3)\}\{a|b,c|d,e,f,g\} \\ = & \pm\{\{x(1|2)\}\{a|- \}\}\{\{y(2|3)\}\{b,c|d,e,f,g\}\} \\ & \pm\{\{x(1|2)\}\{a|- \}\}\{d,\{y(2|3)\}\{b,c|e,f,g\}\}, \end{aligned}$$

and

$$\begin{aligned} & \{x(1|2)\}\{y(2|3)\}\{a|b,c,d|e,f,g\} \\ = & \pm\{\{x(1|2)\}\{a|- \}\}\{\{y(2|3)\}\{b,c|e,f,g\},d\} \\ & \pm\{\{x(1|2)\}\{a|- \}\}\{b,\{y(2|3)\}\{c,d|e,f,g\}\}. \end{aligned}$$

Remark 4 (Shortcut) Suppose we are trying to compose $x(\pi)$ and $y(\pi')$, with

$$\pi = (i_1|\dots|i_r) \quad \text{and} \quad \pi' = (j_1|\dots|j_s).$$

Once we compute the product $\pi * \pi'$ and decide which

$$\tilde{\pi} = (k_1|\dots|k_t) \rightarrow \pi * \pi'$$

to go after, we look at $\{a^{(1)}|\dots|a^{(t)}\}$ and check out which consecutive s slots are large enough to accommodate $y(\pi')$. The remaining elements in these slots as well as y will go into one slot of $x(\pi)$, therefore we also have to make sure that the remaining slots and the combination of chosen slots will conform to the type of x before deciding this is one of the feasible combinations.

Example 14 In [2], we defined **higher order differential operators** $\Delta : A \rightarrow A$ for a noncommutative, nonassociative algebra (A, m) to coincide with the commutative and associative case given by Koszul in [14]. A follow-up on the simplification of notation due to multibraces, and a generalization, appeared in [1]. A linear operator Δ is called a differential operator of order $\leq r$ if and only if a certain $(r+1)$ -linear map

$$\Phi_{\Delta}^{r+1}(a_1, \dots, a_{r+1}) \quad (11)$$

is identically zero; we now want to think of (11) as a partitioned map $\Phi_{\Delta}^{r+1}(r|1)$, because the last slot is distinguished and the original inductive definition

$$\begin{aligned} \Phi_{\Delta}^1(a) &= \Delta(a) \\ \Phi_{\Delta}^2(a, b) &= \Delta(ab) - \Delta(a)b - (-1)^{|\Delta||a|}a\Delta(b) \\ \Phi_{\Delta}^{r+2}(a_1, \dots, a_r, b, c) &= \Phi_{\Delta}^{r+1}(a_1, \dots, a_r, bc) - \Phi_{\Delta}^{r+1}(a_1, \dots, a_r, b)c \\ &\quad - (-1)^{|b|(|\Delta|+|a_1|+\dots+|a_r|)}b\Phi_{\Delta}^{r+1}(a_1, \dots, a_r, c) \quad r \geq 1 \end{aligned}$$

in [2] can be conveniently rewritten as

$$\begin{aligned} \phi_{\Delta}^1(a) &= \{\Delta\}\{a\} \\ \Phi_{\Delta}^2(a, b) &= [\Phi_{\Delta}^1(1), m(2)]\{a, b\} \\ \Phi_{\Delta}^{r+2}(a_1, \dots, a_r, b, c) &= [\Phi_{\Delta}^{r+1}(r|1), m(2)]\{a_1, \dots, a_r|b, c\} \quad r \geq 1 \end{aligned}$$

in terms of Gerstenhaber brackets and partitioned maps. Note that

$$(r|1) * (2) = (2) * (r|1) = (r+1|1) + (r|2).$$

Here we may designate the same multilinear map Φ_{Δ}^{r+2} to be of type $(r+2)$, $(r+1|1)$, or $(r|2)$, depending on what we want to describe! More in Section 4.3.

2.3.2 Higher products of regular partitions

The algebra \mathcal{P} enjoys higher products

$$N(1|\lambda_1|\dots|\lambda_t) : \mathcal{P} \otimes \mathcal{P}^{\otimes \lambda_1} \otimes \dots \otimes \mathcal{P}^{\otimes \lambda_t} \rightarrow \mathcal{P}, \quad (12)$$

where $(\lambda_1|\dots|\lambda_t)$ itself is any regular partition, and

$$N(1|1)(\pi, \pi') \stackrel{\text{def}}{=} \pi * \pi'.$$

Such products will be used to model higher compositions of partitioned maps in the next section.

We had better define

$$\{N(1|k)\}\{(i)|(j_1), \dots, (j_k)\} \stackrel{\text{def}}{=} (i + j_1 + \dots + j_k - k),$$

because we will obtain an $(i + j_1 + \dots + j_k - k)$ -linear map when we compute $\{x(i)\}\{y_1(j_1), \dots, y_k(j_k)\}$: we simply add up all d -gradings. Similarly, it makes sense to define

$$\{N(1|1|\dots|1)\}\{(i)|(j_1)|\dots|(j_k)\} \stackrel{\text{def}}{=} (i + j_1 + \dots + j_k - k),$$

because again

$$d(\{x\}\{y_1\} \dots \{y_k\}) = d(x) + d(y_1) + \dots + d(y_k).$$

Note that \bar{d} is preserved as well in both cases.

Next, we define for $i \geq 2$

$$\begin{aligned} & \{N(1|2)\}\{(i)|(j_1|\cdots|j_s), (k_1|\cdots|k_t)\} \\ \stackrel{\text{def}}{=} & \sum_{u_1+\cdots+u_{s+t-1}=i-2, u_l \geq 0} (j_1+u_1|\cdots|j_{s-1}+u_{s-1}|j_s+k_1+u_s|k_2+u_{s+1}|\cdots|k_t+u_{s+t-1}), \end{aligned}$$

that is, we write down the partition

$$(j_1|\cdots|j_s+k_1|\cdots|k_t),$$

and add to the slots nonnegative integers with total $i-2$ in every possible way. In case of

$$\{N(1|3)\}\{(i)|(j_1|\cdots|j_s), (k_1|\cdots|k_t), (l_1|\cdots|l_u)\} \quad (i \geq 3),$$

we start with

$$(j_1|\cdots|j_{s-1}|j_s+k_1|k_2|\cdots|k_{t-1}|k_t+l_1|l_2|\cdots|l_u),$$

and this time distribute a sum of $i-3$. The generalization to

$$\{N(1|k)\}\{(i)|\pi_1, \dots, \pi_k\}, \quad \pi_l \in \mathcal{B}$$

is clear. Once more, both d and \bar{d} are preserved. In the next section we will see the definition of the composition $\{x\}\{y, z\}$ which motivates the above formulas.

As for

$$\{N(1|1|\cdots|1)\}\{(i)|\pi_1|\cdots|\pi_k\},$$

we recall the meaning of $\{x\}\{y\}\{z\}$ for non-partitioned maps from Section 2.1 (see Example 1):

$$\{x\}\{y\}\{z\} = \{x\}\{y, z\} \pm \{x\}\{z, y\} \pm \{x\}\{\{y\}\{z\}\}.$$

In other words, y and z can be substituted into x separately (and in any order), or z can go into y first. Then by analogy we want to have

$$\begin{aligned} & \{N(1|1|1)\}\{(i)|\pi_1|\pi_2\} \\ \approx & \{N(1|2)\}\{(i)|\pi_1, \pi_2\} + \{N(1|2)\}\{(i)|\pi_2, \pi_1\} + \{N(1|1)\}\{(i)|\{N(1|1)\}\{\pi_1|\pi_2\}\}, \end{aligned}$$

where the symbol \approx again means that any repetition of basis elements $\tilde{\pi}$ on the right is to be ignored. The formula should hold even when the first partition (i) is replaced by any $\pi \in \mathcal{B}$. In fact, all higher products (12) should be constructed from lower products by thinking in terms of the corresponding compositions.

Example 15 *We have*

$$\begin{aligned} & \{x\}\{y_1, y_2\}\{z\} \\ = & \{x\}\{y_1, y_2, z\} \pm \{x\}\{y_1, z, y_2\} \pm \{x\}\{z, y_1, y_2\} \\ & \pm \{x\}\{\{y_1\}\{z\}, y_2\} \pm \{x\}\{y_1, \{y_2\}\{z\}\}, \end{aligned}$$

therefore by definition

$$\begin{aligned} & \{N(1|2|1)\}\{\pi|\pi_1, \pi_2|\pi'\} \\ \approx & \{N(1|3)\}\{\pi|\pi_1, \pi_2, \pi'\} + \{N(1|3)\}\{\pi|\pi_1, \pi', \pi_2\} + \{N(1|3)\}\{\pi|\pi', \pi_1, \pi_2\} \\ & + \{N(1|2)\}\{\pi|\{N(1|1)\}\{\pi_1|\pi'\}, \pi_2\} + \{N(1|2)\}\{\pi|\pi_1, \{N(1|1)\}\{\pi_2|\pi'\}\}. \end{aligned}$$

To complete the discussion of higher products, we replace (i) by $\pi = (i_1|\cdots|i_r)$: then by definition,

$$N(1|\lambda_1|\cdots|\lambda_t)\{\pi|\cdots\} \approx \sum_{\alpha=1}^r (i_1|\cdots|i_{\alpha-1}|\{N(1|\lambda_1|\cdots|\lambda_t)\}\{(i_\alpha)|\cdots|i_{\alpha+1}|\cdots|i_r\}).$$

Example 16

$$\begin{aligned}
& \{N(1|2)\}\{(2|3)|(2), (3|4)\} \\
\approx & (\{N(1|2)\}\{(2)|(2), (3|4)\}|3) + (2|\{N(1|2)\}\{(3)|(2), (3|4)\}) \\
= & (5|4|3) + (2|5|5) + (2|6|4).
\end{aligned}$$

2.3.3 Higher compositions of regular partitioned maps

Following [10] and [1], we would like to define

$$\{x(\pi)\}\{y_1(\pi_1), \dots, y_k(\pi_k)\},$$

and ultimately

$$\{x(\pi)\}\{y_1(\pi_1), \dots, y_k(\pi_k)\} \cdots \{z_1(\pi'_1), \dots, z_l(\pi'_l)\}$$

for $\pi, \pi_i, \pi'_j \in \mathcal{B}$. The rules will be similar to the nonpartitioned case, and types of maps will be governed by the enriched algebra of partitions discussed in the previous section. To give an easy example, we consider

$$\{x(i)\}\{y(\pi_1), z(\pi_2)\}, \quad \text{with } \pi_1 = (j_1 | \cdots | j_s), \pi_2 = (k_1 | \cdots | k_t) \in \mathcal{B}.$$

We choose

$$\tilde{\pi} \rightarrow \{N(1|2)\}\{(i)|\pi_1, \pi_2\}$$

and set up variables

$$\{a^{(1)} | \cdots | a^{(s+t-1)}\}$$

accordingly. Then

$$\{x(i)\}\{y(\pi_1), z(\pi_2)\}\{a^{(1)} | \cdots | a^{(s+t-1)}\} \quad (13)$$

will be defined as a sum over subdivisions S of these variables. Each S will look like

$$\begin{aligned}
a^{(1)} &= \{b^{(1)}, c^{(1)}, d^{(1)}\} \\
&\vdots \\
a^{(s-1)} &= \{b^{(s-1)}, c^{(s-1)}, d^{(s-1)}\} \\
a^{(s)} &= \{b^{(s)}, c^{(s)}, d^{(s)} = e^{(1)}, f^{(1)}, g^{(1)}\} \\
a^{(s+1)} &= \{e^{(2)}, f^{(2)}, g^{(2)}\} \\
&\vdots \\
a^{(s+t-1)} &= \{e^{(t)}, f^{(t)}, g^{(t)}\},
\end{aligned}$$

where $c^{(l)}$ has j_l consecutive elements, $f^{(l)}$ has k_l consecutive elements, and the strings $b^{(l)}, d^{(l)}, e^{(l)}, g^{(l)}$ may be empty. Then the contribution of S to (13) will be

$$\begin{aligned}
\pm \{x(i)\} & \quad \{ \{b^{(1)}\} \cdots \{b^{(s)}\}, \{y(\pi_1)\}\{c^{(1)} | \cdots | c^{(s)}\}, \{d^{(1)}\} \cdots \{d^{(s-1)}\} \{e^{(1)}\} \cdots \{e^{(t)}\}, \\
& \quad \{z(\pi_2)\}\{f^{(1)} | \cdots | f^{(t)}\}, \{g^{(1)}\} \cdots \{g^{(t)}\} \}.
\end{aligned}$$

Any multiple compositions involving more than two pairs of braces can be handled as sums and iterations of the latter: for example,

$$\{x(\pi)\}\{y(\pi_1)\}\{z(\pi_2)\}$$

means

$$\{x(\pi)\}\{y(\pi_1), z(\pi_2)\} \pm \{x(\pi)\}\{z(\pi_2), y(\pi_1)\} \pm \{x(\pi)\}\{\{y(\pi_1)\}\{z(\pi_2)\}\}.$$

2.3.4 Partitioned Hochschild complex

Let us denote the vector space of n -linear maps on A ($n \geq 0$) with values in A by

$$Hom(A^{\otimes n}; A)$$

and the vector space of multilinear maps on A of type $\pi = (i_1 | \dots | i_r) \in \mathcal{B} \cup \{(0)\}$ by

$$Hom(A^\pi; A) = Hom(A^{\otimes i_1} \otimes \dots \otimes A^{\otimes i_r}; A)$$

(the two spaces are clearly isomorphic if $d(\pi) = n - 1$). Depending on whether we want infinite sums or not, we defined the regular Hochschild complex $C^\bullet(A)$ to be either of the spaces

$$\oplus_{n \geq 0} Hom(A^{\otimes n}; A) \subset Hom(\oplus_{n \geq 0} A^{\otimes n}; A) = Hom(TA; A).$$

In a similar manner, we may define the **partitioned Hochschild complex** $C_{\mathcal{P}}^\bullet(A)$ by either of

$$\oplus_{\pi \in \mathcal{B} \cup \{(0)\}} Hom(A^\pi; A) \subset Hom(\oplus_{\pi \in \mathcal{B} \cup \{(0)\}} A^\pi; A) = Hom(T(TA); A).$$

Proposition 2 (Gerstenhaber) *The truncated Hochschild complex*

$$\bar{C}^\bullet(A) = \oplus_{n \geq 1} Hom(A^{\otimes n}; A)$$

is a right pre-Lie algebra under the multiplication

$$x \circ y = \{x\}\{y\}.$$

Proof. See [9], [1], and Proposition 3 below. \square

The following Proposition answers a natural question:

Proposition 3 *The truncated partitioned Hochschild complex*

$$\bar{C}_{\mathcal{P}}^\bullet(A) = \oplus_{\pi \in \mathcal{B}} Hom(A^\pi; A)$$

is a right pre-Lie algebra under the composition of partitioned maps.

Proof. The triple composition

$$\{ \{x(\pi)\} \{y(\pi')\} \} \{z(\pi'')\} - \{x(\pi)\} \{ \{y(\pi')\} \{z(\pi'')\} \}$$

consists of partitioned maps $w(\tilde{\pi})$ obtained by substituting y and z separately into x (we subtract the terms where z goes into y). Then by symmetry

$$\{ \{x(\pi)\} \{z(\pi'')\} \} \{y(\pi')\} - \{x(\pi)\} \{ \{z(\pi'')\} \{y(\pi')\} \}$$

has the same summands up to an overall sign. \square

Remark 5 *The result holds for the full complex in both cases, keeping in mind that $\{a\}\{b_1, \dots, b_n\} = 0$ for $a, b_i \in A$.*

Proposition 4 *The full extended Hochschild complex*

$$C^{\bullet, \bullet}(A) = Hom(\oplus_{i \geq 0} A^{\otimes i}; \oplus_{j \geq 0} A^{\otimes j}) \subset Hom(TA; TA)$$

defined in [1] is a right pre-Lie algebra under the multiplication

$$\{x\}\{y\}.$$

Proof. Same as above. The difference is that now *all* compositions $\{x\}\{y\}$ have a natural definition in the new complex, even when x and y are elements of TA (see Example 8 in Section 2.1). \square

2.3.5 Partitioned maps involving zeros

We now need a theory of compositions of maps of type π for $\pi \in \bar{\mathcal{B}}$, consistent with our earlier definitions. It is clear that $x(0)$ (with $d(x) = -1$) has to be an element of A .

Example 17 For $n \geq 2$, we have

$$(n) * (0) = (n-1),$$

and not surprisingly,

$$\{x(n)\}\{y(0)\}\{a_1, \dots, a_{n-1}\} = \{x(n)\}\{b\}\{a_1, \dots, a_{n-1}\}$$

if $y(0) = b$.

On the other hand,

Example 18 The expression $\{x(0)\}\{y(0)\}$ must be interpreted as zero in $C_{\mathcal{P}}^{\bullet}(A)$, as in $C^{\bullet}(A)$ (after all, $(0) * (0) = 0$). It can be thought of as an element

$$a \otimes b - (-1)^{|a||b|} b \otimes a \in A^{\otimes 2}$$

inside

$$C^{\bullet, \bullet}(A) = \text{Hom}(TA; TA)$$

(or its suitably defined counterpart $C_{\mathcal{P}}^{\bullet, \bullet}(A)$) only if we enlarge the Hochschild complex. Similarly, the composition $\{y(0)\}\{x(n)\}$ for $n \geq 1$ is zero in $C_{\mathcal{P}}^{\bullet}(A)$.

We need only define the subdivisions S corresponding to some

$$\{x(i)\}\{y(j_1 | \dots | j_s)\}\{a^{(1)} | \dots | a^{(s)}\} \quad \text{with } i \geq 1.$$

We again subdivide elements in each slot according to (10), but this time the number of elements in $c^{(l)}$ will be zero if $j_l = 0$. Since $b^{(l)}$ and $d^{(l)}$ were already allowed to be empty, this is not a novel concept!

Example 19 We have

$$(4) * (1|0) = (1|3) + (2|2) + (3|1) + (4|0).$$

Then the six subdivisions for $(2|2)$, or $\{a, b|c, d\}$, will be

$$\begin{array}{l|l} \emptyset, \{a\}, \{b\} & \emptyset, \emptyset, \{c, d\} \\ \emptyset, \{a\}, \{b\} & \{c\}, \emptyset, \{d\} \\ \emptyset, \{a\}, \{b\} & \{c, d\}, \emptyset, \emptyset \\ \{a\}, \{b\}, \emptyset & \emptyset, \emptyset, \{c, d\} \\ \{a\}, \{b\}, \emptyset & \{c\}, \emptyset, \{d\} \\ \{a\}, \{b\}, \emptyset & \{c, d\}, \emptyset, \emptyset, \end{array}$$

and we will have

$$\begin{aligned} & \{x(4)\}\{y(1|0)\}\{a, b|c, d\} \\ = & \{x(4)\}\{ \{y(1|0)\}\{a\}, \{b\}\{c, d\} \} \\ & \pm \{x(4)\}\{c, \{y(1|0)\}\{a\}, \{b\}\{d\} \} \\ & \pm \{x(4)\}\{c, d, \{y(1|0)\}\{a\}, b\} \\ & \pm \{x(4)\}\{a, \{y(1|0)\}\{b\}, c, d\} \\ & \pm \{x(4)\}\{ \{a\}\{c\}, \{y(1|0)\}\{b\}, d\} \\ & \pm \{x(4)\}\{ \{a\}\{c, d\}, \{y(1|0)\}\{b\} \} \end{aligned}$$

as the $(2|2)$ -component of the composition.

Example 20 For $i \geq 2$, the sum

$$\sum_{j_1 + \dots + j_s = 1} \{x(i)\} \{y(j_1 | \dots | j_s)\} \{a^{(1)} | \dots | a^{(s)}\}$$

with each $y(j_1 | \dots | j_s)$ as the (linear) identity map on A gives us exactly

$$i \{x(i)\} \{a^{(1)}\} \dots \{a^{(s)}\}.$$

In particular,

$$\{x(i)\} \{\text{id}(1)\} \{a_1, \dots, a_i\} = i \{x(i)\} \{a_1, \dots, a_i\}.$$

3 The master identity for homotopy Gerstenhaber algebras

The present work on the master identity for G_∞ algebras originated from a joint paper of Kimura, Voronov, and Zuckerman [13], where a nontrivial algebra over a certain cellular operad $K_\bullet \underline{\mathcal{M}}$ (first defined by Getzler and Jones [11], based on ideas of Fox-Neuwirth) is described. Since [13] is an excellent introduction to the subject and contains an in-depth description of $K_\bullet \underline{\mathcal{M}}$, we will skip the origins and details of the following definition:

Definition 1 A homotopy Gerstenhaber algebra (G_∞ algebra) is an algebra over the operad $K_\bullet \underline{\mathcal{M}}$.

Instead, in this section, we will expand the discussion in [13] of how the individual G_∞ identities are obtained from certain “configurations”. Our explanations will not require any background, and the pictures will eventually coagulate into the equivalent

Definition 2 A homotopy Gerstenhaber algebra (G_∞ algebra) is a super graded vector space

$$A = \bigoplus_{n \in \mathbb{Z}} A^n$$

together with a collection

$$m(\pi), \quad |m(\pi)| \equiv d(\pi) + \bar{d}(\pi) + 1 \pmod{2}$$

of partitioned multilinear maps, such that

$$m(1|0) = m(0|1) = \text{id}_A$$

are the only nonzero ones among $m(\pi)$ with $\pi \in \mathcal{B}_0$, where the formal sum

$$m = \sum_{\pi \in \mathcal{B}} m(\pi)$$

satisfies

$$\{\tilde{m}\} \{\tilde{m}\} = 0. \tag{14}$$

Remark 6 Identity (14) means the finite sum of partitioned maps of type $\tilde{\pi}$ is equal to zero in the composition for each partition $\tilde{\pi}$. It is understood that if a map $m(\tilde{\pi})$ is identically zero, then we do NOT consider the $\tilde{\pi}$ part of (14) among the G_∞ subidentities. The Hochschild complex in Section 4.2 is a good example. Also, we define the **total degree** $||m(\pi)||$ to be

$$||m(\pi)|| \stackrel{\text{def}}{=} d(\pi) + \bar{d}(\pi) + |m|,$$

which is always odd in a G_∞ algebra. The actual super degree comes from dimensions of chains in the G_∞ operad $K_\bullet \underline{\mathcal{M}}$.

We emphasize again that the elegant notation of [13] has to be changed in order to reserve the older notation of braces for compositions of maps. The symbol $m(i_1 | \dots | i_r)$ will be used to denote the multilinear map in [13] shown by r pairs of braces with i_1, \dots, i_r arguments respectively.

The configurations which “contribute to the differential” and hence show up in the G_∞ algebra identities are as follows: for each regular ordered partition

$$\tilde{\pi} = (k_1 | \dots | k_t) \in \mathcal{B}$$

we draw t vertical lines with k_1, \dots, k_t points on them respectively from left to right, labelled by

$$\{a_1, \dots, a_{k_1 + \dots + k_t}\} = \{a^{(1)} | \dots | a^{(t)}\}$$

(the lexicographical ordering is from top to bottom and left to right). We then make a list of all possible configurations in which one smooth “bubble” surrounds several points in this picture, subject to the following conditions:

- If two points on the same vertical line are in the bubble, so are all points between them.
- If two points on different vertical lines are in the bubble, so is at least one point from each line in between the two lines.

Note that bubbles around only one point and the giant bubble enclosing all points are acceptable. We will call these configurations Type I. Next, we introduce bubbles that contain exactly one point on one line and an empty portion of an adjacent line and call the resulting pictures Type II. Bubbles with empty portions positioned between different points will be counted as different (the empty portion may also be above or below all points on the second line). If we start with only one vertical line, there will be no Type II pictures.

Type I pictures translate into the quadratic parts of the lower identities given in [13] and those of Type II into the seemingly non-quadratic parts. The goal in the Type I situation is to collect together all pictures which depict the composition of $m(\pi)$ with $m(\pi')$ ($\pi, \pi' \in \mathcal{B}$) such that $\tilde{\pi} \rightarrow \pi * \pi'$. Each picture is really one subdivision S for some $\{m(\pi)\}\{m(\pi')\}$. The original picture without the bubble represents

$$\{m(k_1 | \dots | k_t)\}\{a^{(1)} | \dots | a^{(t)}\};$$

we understand that each vertical line corresponds to one slot. Similarly, the inside of a bubble is what we would have denoted in Section 2.3.1 by

$$\{m(j_1 | \dots | j_s)\}\{c^{(1)} | \dots | c^{(k)}\}$$

for a particular subdivision S of $\{a^{(1)} | \dots | a^{(t)}\}$, sitting as an element of A inside some $m(i_1 | \dots | i_r)$. Any vertical lines seen above the bubble will have the leftover elements which had formerly been denoted by

$$b^{(1)}, \dots, b^{(k)}$$

and the ones below will have the elements

$$d^{(1)}, \dots, d^{(k)}.$$

We transcribe each of these pictures into symbols and add up!

In order to complete the $\tilde{\pi}$ portion of the identity $\{\tilde{m}\}\{\tilde{m}\} = 0$ we introduce a picture containing two vertical lines with one point on the left (resp. right) and no points on the right (resp. left) to mean $m(1|0)$ (resp. $m(0|1)$). By Example 20 we know that the overall contribution of Type II pictures for $\tilde{\pi} = (k_1 | \dots | k_t)$ will be

$$\sum_{\alpha=1}^{t-1} (k_\alpha + k_{\alpha+1}) \quad \{m(k_1 | \dots | k_{\alpha-1} | k_\alpha + k_{\alpha+1} | k_{\alpha+2} | \dots | k_t)\} \\ \{a^{(1)} | \dots | a^{(\alpha-1)} | \{a^{(\alpha)}\}\{a^{(\alpha+1)}\} | a^{(\alpha+2)} | \dots | a^{(t)}\}.$$

Finally, we set the sum of all Type I and Type II terms equal to zero.

Example 21 For example, two vertical lines with points $\{a\}$ and $\{b, c\}$ respectively (i.e. the original picture for $\tilde{\pi} = (1|2)$) give us seven Type II pictures, which will add up to

$$3\{m(3)\}\{a\}\{b, c\}.$$

Then the overall subidentity for $(1|2)$ is

$$\begin{aligned} & \{m(1)\}\{m(1|2)\}\{a|b, c\} \pm \{m(1|2)\}\{m(1)\}\{a|b, c\} \\ & \pm \{m(2)\}\{\{m(1|1)\}\{a|b\}, c\} \pm \{m(2)\}\{b, \{m(1|1)\}\{a|c\}\} \\ & \pm \{m(1|1)\}\{a|\{m(2)\}\{b, c\}\} \pm 3\{m(3)\}\{a\}\{b, c\} = 0. \end{aligned}$$

This is Eqn. (8) of [13] with the exception of the coefficient 3; see Remark 7 below. To check that all degrees are preserved, one must take into account $\bar{d}(\{a|b, c\}) = 1$ and $\bar{d}(\{a\}\{b, c\}) = 0$.

Example 22 Let us write down all terms of $\{\tilde{m}\}\{\tilde{m}\} = 0$ which correspond to the partition $\tilde{\pi} = (1|2|3)$. We start with a picture having three vertical lines, with points $\{a\}$, $\{b, c\}$, and $\{d, e, f\}$ respectively, from left to right. Here is a list of all bubbles in Type I pictures and the terms they contribute (up to sign):

bubbles in Type I pictures	relevant product $\pi * \pi' = (1 2 3) + \dots$	contributed terms (up to sign)
abcdef	$(1) * (1 2 3)$	$\{m(1)\}\{m(1 2 3)\}\{a b, c d, e, f\}$
a, b, c, d, e, f	$(1 2 3) * (1)$	$\{m(1 2 3)\}\{m(1)\}\{a b, c d, e, f\}$
abdef, acdef	$(2) * (1 1 3)$	$\{m(2)\}\{m(1 1 3)\}\{a b, c d, e, f\}$
bc	$(1 1 3) * (2)$	$\{m(1 1 3)\}\{m(2)\}\{a b, c d, e, f\}$
abcde, abcef	$(2) * (1 2 2)$	$\{m(2)\}\{m(1 2 2)\}\{a b, c d, e, f\}$
de, ef	$(1 2 2) * (2)$	$\{m(1 2 2)\}\{m(2)\}\{a b, c d, e, f\}$
abde, abef, acde, acef	$(3) * (1 1 2)$	$\{m(3)\}\{m(1 1 2)\}\{a b, c d, e, f\}$
abcd, abce, abcf	$(3) * (1 2 1)$	$\{m(3)\}\{m(1 2 1)\}\{a b, c d, e, f\}$
def	$(1 2 1) * (3)$	$\{m(1 2 1)\}\{m(3)\}\{a b, c d, e, f\}$
abd, abe, abf, acd, ace, acf	$(4) * (1 1 1)$	$\{m(4)\}\{m(1 1 1)\}\{a b, c d, e, f\}$
bcdef	$(1 1) * (2 3)$	$\{m(1 1)\}\{m(2 3)\}\{a b, c d, e, f\}$
ab, ac	$(2 3) * (1 1)$	$\{m(2 3)\}\{m(1 1)\}\{a b, c d, e, f\}$
bdef, cdef	$(1 2) * (1 3)$	$\{m(1 2)\}\{m(1 3)\}\{a b, c d, e, f\}$
abc, bde, bef, cde, cef	$(1 3) * (1 2)$	$\{m(1 3)\}\{m(1 2)\}\{a b, c d, e, f\}$
bcde, bcef	$(1 2) * (2 2)$	$\{m(1 2)\}\{m(2 2)\}\{a b, c d, e, f\}$
bcd, bce, bcf	$(1 3) * (2 1)$	$\{m(1 3)\}\{m(2 1)\}\{a b, c d, e, f\}$
bd, be, bf, cd, ce, cf	$(1 4) * (1 1)$	$\{m(1 4)\}\{m(1 1)\}\{a b, c d, e, f\}$

It can be checked that all products $\pi * \pi'$ such that $(1|2|3) \rightarrow \pi * \pi'$ are in the above list, and for each such product every subdivision S leading to $(1|2|3)$ is given by a bubble! Next, we list the two Type II pictures and their contributions:

bubbles in Type II pictures	relevant product $\pi * \pi' = (1 2 3) + \dots$	contributed terms (up to sign)
$a\emptyset$ (three)	$(3 3) * (1 0)$	$3\{m(3 3)\}\{\{a\}\{b, c\} d, e, f\}$
$\emptyset b, \emptyset c$ (two each)	$(3 3) * (0 1)$	
$b\emptyset, c\emptyset$ (four each)	$(1 5) * (1 0)$	$5\{m(1 5)\}\{a \{b, c\}\{d, e, f\}\}$
$\emptyset d, \emptyset e, \emptyset f$ (three each)	$(1 5) * (0 1)$	

We add up all contributed terms, expanding the ones in the first table according to the instructions in Section 2.3.1, and set the sum equal to zero.

Examples of lower identities (up to three points in all) can be found in [13].

Remark 7 *The coefficients in Type II are a major nuisance and are not a part of the G_∞ identities introduced in [13]. In order to get rid of them one may sacrifice consistency in the definition of composition as a whole and define Type II as $t - 1$ pictures in which an empty bubble is skewered to the (say) bottom of any two adjacent lines; then the two lines protruding above will be treated as if the bubble is full; i.e. they will be intertwined into one etc. and we will have the “non-quadratic” part without the coefficient. Another way out would be to define a new unary operation, say F , on the partitioned maps which describes what happens when adjacent strands are intertwined two by two, and write the master equation as*

$$\{\tilde{m}\}\{\tilde{m}\} + F(\tilde{m}) = 0,$$

where the first summand comes from Type I pictures only, and F has total degree 1 (mod 2).

4 Substructures and examples

4.1 Substructures

Let (A, m) be a G_∞ algebra as in Definition 2. Note that if desired, we may modify this algebraic definition to make any subset of $\{m(\pi)\}_{\pi \in \mathcal{B}}$, especially of $\{m(\pi)\}_{\pi \in \mathcal{B}_0}$, vanish.

Lemma 1 *The infinite sum*

$$m_A = m(0) + m(1) + m(2) + \cdots$$

satisfies the relation

$$\{\tilde{m}_A\}\{\tilde{m}_A\} = 0. \tag{15}$$

Therefore, (A, m_A) is an A_∞ algebra and (A, l_A) is an L_∞ algebra, where l_A is the term-by-term graded antisymmetrization of m_A .

Proof. The identity (15) is equivalent to

$$\{\{\tilde{m}\}\{\tilde{m}\}\}(i) = 0 \quad \forall i \geq 0,$$

hence is valid, for the following reasons: the span of $\{(i)\}_{i \geq 1}$ is a subalgebra of $(\mathcal{P}, *)$. Moreover, if

$$(i) \rightarrow \pi * \pi',$$

then we must have

$$\bar{d}(\pi) + \bar{d}(\pi') = \bar{d}(i) = 0,$$

or

$$\bar{d}(\pi) = \bar{d}(\pi') = 0. \square$$

Lemma 2 *The infinite sum*

$$m_B = m(1) + m(1|1) + m(1|1|1) + \cdots$$

satisfies

$$\{\tilde{m}_B\}\{\tilde{m}_B\} = 0, \tag{16}$$

provided that all $\{m(\pi)\}_{\pi \in \mathcal{B}_0}$ vanish. Thus (A, m_B) is an A_∞ algebra and (A, l_B) is an L_∞ algebra, where l_B is the graded antisymmetrization of m_B .

Proof. This time (16) is equivalent to

$$\{ \{ \tilde{m} \} \{ \tilde{m} \} \} \pi_t = 0 \quad \forall \pi_t = (1|1| \cdots |1), d(\pi_t) = \bar{d}(\pi_t) = t - 1.$$

To see why, we note that the span of $\{\pi_t\}_{t \geq 1}$ is again a subalgebra of \mathcal{P} , and whenever

$$\pi_t \rightarrow \pi * \pi',$$

we have

$$d(\pi) + d(\pi') = \bar{d}(\pi) + \bar{d}(\pi') = t - 1$$

and

$$d(\pi) \geq \bar{d}(\pi), \quad d(\pi') \geq \bar{d}(\pi') \quad (\text{always true for } \pi, \pi' \in \mathcal{B}),$$

implying

$$d(\pi) = \bar{d}(\pi) \quad \text{and} \quad d(\pi') = \bar{d}(\pi'). \square$$

In [13], (A, m_B) in a topological vertex operator algebra A (more generally, in a G_∞ algebra) is said to give rise to an L_∞ algebra in the graded antisymmetrization. In fact, m_B misses being A_∞ itself because of the nonzero maps $m(1|0)$ and $m(0|1)$: we have

$$\pi_t \rightarrow \pi * \pi' \quad \text{with } \pi = (1|1| \cdots |2| \cdots |1|1) \text{ and } \pi' = (1|0) \text{ or } (0|1),$$

in addition to the products

$$\pi_r * \pi_s = \pi_{r+s-1}.$$

4.2 Hochschild complex revisited

Recall that the truncated Hochschild complex $\bar{C}^\bullet(A)$ is a right pre-Lie algebra under the multiplication

$$M(1|1)(x, y) \stackrel{\text{def}}{=} \{x\}\{y\},$$

regardless of any structure on the vector space A (Proposition 2). Moreover, if (A, m) is an associative algebra, then the **differential**

$$M(1)(x) \stackrel{\text{def}}{=} [m, x],$$

dot product

$$M(2)(x, y) \stackrel{\text{def}}{=} \pm \{m\}\{x, y\},$$

and partitioned maps

$$M(1|n)(x, y_1, \dots, y_n) \stackrel{\text{def}}{=} \pm \{x\}\{y_1, \dots, y_n\} \quad n \geq 1$$

are known to satisfy the G_∞ identities ([12],[13]); this was first noticed by Gerstenhaber and Voronov and communicated to Getzler. We provide a brief algebraic proof, in the light of the new terminology.

Proposition 5 (Gerstenhaber-Voronov-Getzler-Jones) *The truncated Hochschild complex*

$$(\bar{C}^\bullet(A), M(1), M(2), \{M(1|n)\}_{n \geq 1})$$

is a G_∞ algebra.

Proof. We have

$$\begin{aligned} (1) * (1) &= (1) \\ (1) * (2) &= (2) * (1) = (2) \\ (2) * (2) &= (3) \\ (1) * (1|n) &= (1|n) * (1) = (1|n) \\ (2) * (1|n) &= (1|n) * (2) = (1|n+1) + (2|n) \\ (1|n) * (1|m) &= \text{sum of triple-slot terms} \end{aligned}$$

as the only products involving (1), (2), or (1|n) on either side and leading to a nontrivial identity. But by Remark 6 we need only check

- (i) $M(1)^2 = 0$
- (ii) $\{M(1)\}\{M(2)\} \pm \{M(2)\}\{M(1)\} = 0$
- (iii) $(\{M(1)\}\{M(1|n+1)\} \pm \{M(1|n+1)\}\{M(1)\} \pm \{M(2)\}\{M(1|n)\} \pm \{M(1|n)\}\{M(2)\})(1|n+1) = 0,$

and not

- (iv) $\{M(2)\}\{M(2)\} = 0,$
- (v) $(\{M(2)\}\{M(1|n)\} \pm \{M(1|n)\}\{M(2)\})(2|n) = 0,$

nor the triple-slot identities. The first two tell us that the differential is square-zero and is a derivation of the dot product; these well-known facts are in [9]. The third is Eqn. (4) of [13] and Eqn. (8) of [22] in disguise. Although not strictly a G_∞ identity, Eqn. (iv) happens to hold in the complex (the dot product is associative) but Eqn. (v) doesn't (it looks like Eqn. (3) in [13] but contains fewer terms). Short proofs of (i)-(iv) are also in [1]. \square

Remark 8 *Once again, the result holds for the full complex with all $\{a\}\{b_1, \dots, b_n\} = 0$ for $a, b_i \in A$. It would also be interesting to study the cases where A is an A_∞ algebra ([10], [1]) and/or $\bar{C}^\bullet(A)$ is replaced by $C^{\bullet, \bullet}(A)$ with the additional composition maps $M(\lambda)$ [2].*

4.3 Phi-operators revisited

The multilinear operators Φ_Δ^r defined at the end of Section 2.3.1 may be a convenient tool to construct brand new G_∞ algebras starting with a small data set. Let us consider an easy truncated version: define

$$\begin{aligned}\Phi(1) &\stackrel{\text{def}}{=} \Phi_\Delta^1 = \Delta \\ \Phi(2) &= \Phi(1|1) \stackrel{\text{def}}{=} \Phi_\Delta^2 \\ \Phi(1|2) &\stackrel{\text{def}}{=} \Phi_\Delta^3,\end{aligned}$$

where Δ is any odd square-zero operator on an algebra (A, m) (m not necessarily commutative or associative). This truncation especially makes sense when Δ is a differential operator of order three. The relevant products in \mathcal{P} are

$$\begin{aligned}(1) * (1) &= (1) \\ (1) * (2) &= (2) * (1) = (2) \\ (1) * (1|1) &= (1|1) * (1) = (1|1) \\ (1) * (1|2) &= (1|2) * (1) = (1|2) \\ (2) * (1|1) &= (1|1) * (2) = (1|2) + (2|1),\end{aligned}$$

and we want

- (i) $\Delta^2 = 0$
- (ii) Δ is a derivation of Φ_Δ^2
- (iii) ditto
- (iv) $(\{\Phi(1)\}\{\Phi(1|2)\} \pm \{\Phi(1|2)\}\{\Phi(1)\} \pm \{\Phi(2)\}\{\Phi(1|1)\} \pm \{\Phi(1|1)\}\{\Phi(2)\})(1|2) = 0.$

Again (i)-(iii) are known (see [2] or [1]), and Eqn. (iv) can be obtained from Lemma 6 of [1]:

Lemma 3 For odd linear operators T and U on A , the Gerstenhaber bracket $[T, U] = TU + UT$ is related to the Gerstenhaber bracket of the Φ operators as follows:

$$\begin{aligned}\Phi_{[T,U]}^1(a) &= [\Phi_T^1, \Phi_U^1]\{a\} \\ \Phi_{[T,U]}^2(a, b) &= [\Phi_T^1, \Phi_U^2]\{a, b\} + [\Phi_U^1, \Phi_T^2]\{a, b\} \\ \Phi_{[T,U]}^3(a, b, c) &= [\Phi_T^1, \Phi_U^3]\{a, b, c\} + [\Phi_U^1, \Phi_T^3]\{a, b, c\} \\ &\quad + [\Phi_T^2, \text{ad}(\Phi_U^2)\{a\}]\{b, c\} + [\Phi_U^2, \text{ad}(\Phi_T^2)\{a\}]\{b, c\}.\end{aligned}$$

Here the adjoint operator is defined as

$$\text{ad}(\Phi_\Delta^r)\{a_1\}\{a_2, \dots, a_r\} \stackrel{\text{def}}{=} \Phi_\Delta^r(a_1, a_2, \dots, a_r).$$

By substituting $T = U = \Delta$ we find that the last equation in the above Lemma is exactly Eqn. (iv), by virtue of $[\Delta, \Delta] = 2\Delta^2 = 0$. Therefore, we have

Proposition 6 If Δ is an odd square-zero operator on an arbitrary algebra (A, m) , then

$$(A, \Phi(1), \Phi(2), \Phi(1|1), \Phi(1|2))$$

is a G_∞ algebra.

4.4 Topological vertex operator algebras

Topological vertex operator algebras (TVOA) have always fueled the subject of G_∞ algebras; see [16] and [13]. The article [13] indicates the existence of a G_∞ structure on a TVOA but does not produce the actual products. We will assume some acquaintance with vertex operator super algebras (VOSA) and TVOA's, which are VOSA's with extra structure ([7], [6], [3], [8], [17], [18], [19]). A **VOSA** is a \mathbf{Z} -bigraded vector space V (one L_0 and one super, or fermionic, grading) in which we associate to every element u a unique **vertex operator** $u(z) = \sum u_n z^{-n-1}$, with $u_n \in \text{End}(V)$. There is an action of the Virasoro algebra by $L(z) = \sum L_n z^{-n-2}$ where the eigenvalues of L_0 are bounded from below and L_{-1} is formal differentiation. The vacuum element $\mathbf{1}$, represented by $1 \cdot z^0$, is the identity element with respect to the distinguished **Wick product** given by $u_{-1}v$. Although the overall identities satisfied by all the bilinear products $u_n v$ are summed up by the formal relation ([DL])

$$[u(z_1), v(z_2)](z_1 - z_2)^t = 0 \quad \text{for sufficiently large } t = t(u, v),$$

the specialized identities

$$(u_m v)_n = \sum_{i \geq 0} (-1)^i \binom{m}{i} (u_{m-i} v_{n+i} - (-1)^{m+|u||v|} v_{m+n-i} u_i)$$

and

$$[u_m, v_n] = \sum_{i \geq 0} \binom{m}{i} (u_i v)_{m+n-i}$$

(for $m, n \in \mathbf{Z}$) are very useful. For the record, we list a number of products already shown to satisfy the G_∞ identities:

- The differential (“BRST operator”), usually denoted by Q , can be taken to be the odd linear operator $m(1)$.
- The Wick product (“normal ordered product”) plays the role of the even bilinear operator $m(2)$ in any VOSA.

- The odd trilinear operation $n(u, v, t)$ of [16] (Eqn. (2.16)) looks like $m(3)$.
- The odd bilinear product $m(u, v)$ of [16] (Eqn. (2.14)) is like $m(1|1)$.

There are additional products in [16] and [2] on a TVOA which may eventually be linked to the G_∞ structure (most of these are related to homotopy Batalin-Vilkovisky algebras).

In addition, we establish a surprising property of the Wick product on a VOSA:

Proposition 7 *Any VOSA V with the Wick product is a left pre-Lie algebra. Therefore, the Wick bracket defined by*

$$[u, v]_W \stackrel{\text{def}}{=} u_{-1}v - (-1)^{|u||v|}v_{-1}u$$

is a graded Lie bracket on V .

Proof. Dropping the super degrees, we have

$$(u_{-1}v)_{-1} = \sum_{i \geq 0} (u_{-1-i}v_{-1+i} + v_{-2-i}u_i) = u_{-1}v_{-1} + \sum_{i \geq 0} (u_{-2-i}v_i + v_{-2-i}u_i),$$

so that for the left (Wick) multiplication operators L_u we have the identity

$$\begin{aligned} & L_u L_v - L_v L_u - L_{[u, v]_W} \\ &= u_{-1}v_{-1} - v_{-1}u_{-1} - (u_{-1}v)_{-1} + (v_{-1}u)_{-1} \\ &= u_{-1}v_{-1} - v_{-1}u_{-1} - u_{-1}v_{-1} - \sum_{i \geq 0} (u_{-2-i}v_i + v_{-2-i}u_i) + v_{-1}u_{-1} + \sum_{i \geq 0} (v_{-2-i}u_i + u_{-2-i}v_i) \\ &= 0. \quad \square \end{aligned}$$

It was pointed out to the author by Haisheng Li that the Jacobi identity for $[u, v]_W$ was independently observed by Dong, Li, and Mason in [4]. The author has also found some evidence (in low L_0 degrees) that the **Zhu product** $*_0$ [23] on a VOSA will turn out to be a left pre-Lie product; a direct proof seems formidable (meanwhile, higher products $*_n$ for $n \geq 1$ as defined by Dong, Li, and Mason in [5] do not seem to be pre-Lie). The Zhu product makes a quotient of any VOSA into an associative algebra, and it is interesting that the **Zhu bracket**

$$[u, v]_Z \stackrel{\text{def}}{=} u *_0 v - (-1)^{|u||v|}v *_0 u$$

may already be Lie on the original space. The pre-Lie identity is also part of the definition of a **Novikov algebra** (see e.g. Osborn's [20]) which has been studied in some detail in terms of the representation theory and also in the context of vertex operator algebras, variational calculus, etc.

Finally, since the modes u_n of vertex operators for $n \geq 0$ have been identified as differential operators of order $n + 1$ with respect to the Wick product, and the Φ operators have been shown to be an invaluable tool in proving identities on VOSA's (such as those related to the generalized Batalin-Vilkovisky structure) ([2],[1]), we expect the Φ operators to play some role in making explicit the G_∞ structure on a TVOA.

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